Ekman Transport in Balanced Currents with Curvature

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ABSTRACT

Ekman transport, the horizontal mass transport associated with a wind stress applied on the ocean surface, is modified by the vorticity of ocean currents, leading to what has been termed the nonlinear Ekman transport. This article extends earlier work on this topic by deriving solutions for the nonlinear Ekman transport valid in currents with curvature, such as a meandering jet or circular vortex, and for flows with the Rossby number approaching unity. Tilting of the horizontal vorticity of the Ekman flow by the balanced currents modifies the ocean response to surface forcing, such that, to leading order, winds parallel to the flow drive an Ekman transport that depends only on the shear vorticity component of the vertical relative vorticity, whereas across-flow winds drive transport dependent on the curvature vorticity. Curvature in the balanced flow field thus leads to an Ekman transport that differs from previous formulations derived under the assumption of straight flows. Notably, the theory also predicts a component of the transport aligned with the surface wind stress, contrary to classic Ekman theory. In the case of the circular vortex, the solutions, extending possible applications of the theory to strong balanced flows. The existence of oscillations, and the potential for resonance and instability, in the Ekman flow at a curved jet are also demonstrated.

1. Introduction

The classic Ekman balance can be understood in terms of vorticity dynamics as a balance between the turbulent diffusion of horizontal vorticity and the tilting of vertical planetary vorticity (Thomas and Rhines 2002). This balance leads to a horizontal mass transport with a magnitude that is simply the ratio of the surface wind stress and the Coriolis frequency, a powerful framework for understanding the influence of surface forcing on the ocean (Ekman 1905). Beyond the implications for horizontal flows and transport, spatial variability in the Ekman transport also generates vertical velocities in the near-surface ocean (Ekman pumping), which provides a boundary condition for the interior flow, central to many theories of the general ocean circulation (Sverdrup 1947; Stommel and Arons 1960; Pedlosky 1979; and references therein).

The seminal works of Stern (1965) and Niiler (1969) extended classic Ekman theory to include the tilting of vertical relative vorticity, a modification often referred to as "nonlinear" Ekman theory. Importantly, contrary

to classic Ekman theory, the inclusion of relative vorticity modifies the Ekman pumping velocity such that a horizontally uniform wind stress can still drive vertical velocities (Stern 1965). Given the relatively large scale of atmospheric motions [O(100) km], compared to the typical scales of ocean dynamics [O(10) km], nonlinearity may be a particularly effective mechanism for generating vertical velocities in the ocean. Consequently, the effect of relative vorticity on Ekman transport and pumping has been the subject of broad interest, and results of both numerical and observational work on this topic suggest that relative vorticity is likely important to the Ekman balance across a range of dynamic processes and scales, modifying both the physics and biology of the upper ocean (Mahadevan and Tandon 2006; Mahadevan et al. 2008; Pedlosky 2008; Gaube et al. 2015).

Despite the recognized importance of these dynamics, existing expressions for the nonlinear Ekman transport were derived under an assumption of straight flows (i.e., to leading order the velocity field is assumed invariant in one direction; Niiler 1969; Thomas and Rhines 2002), and hence they are not applicable to flows with curvature. A more general expression for the vertical Ekman pumping velocity was given by Stern (1965); however,

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this solution, derived using scale analysis of the vorticity equation, is accurate only to first order in Rossby number and cannot be used to determine the horizontal Ekman transport components. These are important limitations both because the effects of relative vorticity on Ekman dynamics scale with the Rossby number and because the horizontal Ekman transport is itself of independent interest. Knowledge of the horizontal Ekman transport is essential for understanding a diverse range of processes, including the frictional flux of potential vorticity (Thomas and Ferrari 2008), Ekman buoyancy flux (Thomas and Lee 2005; Pallàs-Sanz et al. 2010), the energetics of wind-forced symmetric instability (Thomas and Taylor 2010; D'Asaro et al. 2011), mode water variability (Rintoul and England 2002), and the flux of biogeochemical tracers in the surface mixed layer (Franks and Walstad 1997; Williams and Follows 1998; Mahadevan 2016).

In this article, we therefore extend these earlier results on nonlinear Ekman dynamics to provide expressions for the Ekman transport that are valid for balanced flows with curvature and for flows with the Rossby number approaching unity. In section 2, we first summarize earlier theoretical contributions on nonlinear Ekman theory and then derive the equations governing the Ekman transport for an arbitrarily curving balanced current. Analytical solutions for the horizontal transport are found for the case of a circular vortex (section 3), which allows the vertical Ekman pumping velocity to be calculated to a higher order of accuracy than possible with previous solutions. Approximate solutions, valid for weakly nonlinear currents with arbitrary curvature, are given in section 4. These approximate transport solutions are shown to provide the correct expressions for the horizontal transport components associated with the vertical Ekman pumping velocity derived by Stern (1965). Finally, the potential for oscillations, resonance, and growing instabilities in the nonlinear Ekman flow are discussed in section 5.

2. Theory

a. Prior formulations

It is worthwhile to begin with a brief review of important related work on Ekman dynamics in balanced flows with significant relative vorticity. This problem was first considered by Stern (1965), motivated by understanding the effects of a uniform wind stress on a balanced vortex flow. There are two velocity scales for the problem: one associated with the balanced flow U and the other with the Ekman flow, $U_e \equiv \tau_o/(\rho_o f h_e)$, where τ_o is a scale for the wind stress, ρ_o is the density of seawater, f is the Coriolis frequency, and h_e is the Ekman depth. Stern (1965) considered flows where,

$$\varepsilon_e \ll 1 \quad \text{and} \quad \varepsilon \ll 1, \tag{1}$$

where $\varepsilon_e = U_e/fL$ is the Ekman Rossby number, $\varepsilon = U/fL$ is the balanced Rossby number, and L is a characteristic horizontal length scale. Through scale analysis of the vorticity equation, Stern found that, to $O(\varepsilon)$, the Ekman pumping velocity is given by

$$w_{\text{STERN}} \approx \nabla \cdot \frac{\boldsymbol{\tau} \times \hat{\boldsymbol{z}}}{\rho_o(f+\zeta)},$$
 (2)

where τ is the surface wind stress vector, and $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vertical component of the relative vorticity, with *u* and *v* as the zonal and meridional velocity components of the balanced flow, respectively. This solution is notable both for its simplicity and its applicability to flows of any geometry. It is important, however, to emphasize that Stern's result is strictly for the Ekman pumping velocity, and, despite occasional misinterpretation in the literature, it is not generally correct to assume that the horizontal transport, **M**_{STERN}, is given by

$$\mathbf{M}_{\text{STERN}} \neq \frac{\boldsymbol{\tau} \times \hat{\mathbf{z}}}{\rho_o(f + \zeta)}.$$
(3)

Rather, the solution for w_e , in terms of the divergence of a vector field, only constrains the horizontal transport up to a solenoidal vector field. The correct Ekman transport associated with Stern's solution can therefore be written as

$$\mathbf{M}_{\text{STERN}} \approx \frac{\boldsymbol{\tau} \times \hat{\mathbf{z}}}{\rho_o(f + \zeta)} + \nabla \times \mathbf{A}, \qquad (4)$$

where **A** is a vector potential that is not determined in Stern's analysis.

An alternate approach was taken by Niiler (1969), who, interested in applications in the Gulf Stream, solved for the horizontal Ekman transport at a straight jet. The resulting solutions are accurate to a higher order in ε than Stern (1965), and only the nonlinear Ekman self-advection terms in the momentum equations are neglected. The conditions on the validity of Niiler's solution can thus be given as

$$\varepsilon_{e} \ll 1 \quad \text{and} \quad \varepsilon < 1, \tag{5}$$

for a balanced flow that is invariant in one horizontal direction. We note that the condition $\varepsilon < 1$ was not explicitly discussed by Niiler (1969) but is a consequence of the steady-state assumption, which requires that $f(f + \zeta) > 0$ in order to maintain inertial stability

(Holton 2004, p. 205). Given this, Niiler's solution for the horizontal Ekman transport generated by a uniform wind stress over a jet oriented in the north–south direction is

$$\mathbf{M}_{\text{STERN}} \approx \left[\frac{\tau_y}{\rho(f + \partial v/\partial x)}, -\frac{\tau_x}{\rho f}\right].$$
 (6)

It is evident that, on an f plane, the divergence of (6) gives an Ekman pumping velocity consistent with (2).

These earlier contributions thus provide an expression for the vertical Ekman pumping velocity, accurate to $O(\varepsilon)$, and solutions for the horizontal Ekman transport, valid only for straight flows. The solutions given here extend these earlier results by allowing for the calculation of vertical velocities, which are accurate to a higher order in ε , and by providing expressions for the horizontal Ekman transport valid in balanced flows with curvature.

b. Derivation in balanced natural coordinates

Consider a steady current, with a balanced velocity $\overline{\mathbf{u}}$. The total horizontal flow can be written as $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}_e$, where \mathbf{u}_e is the wind-forced Ekman component, ignoring the time dependence and other sources of frictional flow (Wenegrat and McPhaden 2016a,b). It is further assumed that the balanced flow is either barotropic or the Ekman layer is sufficiently thin so as to allow the balanced flow to be approximated as barotropic (i.e., $h_e \ll h$, where h is the depth scale of the balanced flow).

We are interested in finding solutions for the Ekman flow in the presence of a steady, spatially uniform, wind stress. The variables can thus be nondimensionalized as follows: $\overline{u} = U \overline{u}'$, $(u_e, v_e, w_e) = U_e[u'_e, v'_e, (h_e/L)w'_e]$, $(\tau_x, \tau_y) = \tau_o(\tau'_x, \tau'_y)$, x = Lx', y = Ly', $z = h_e z'$, where primes denote nondimensional variables. Using these scalings, the balanced flow is assumed to satisfy

$$\varepsilon \overline{\mathbf{u}}' \cdot \nabla \overline{\mathbf{u}}' + \hat{\mathbf{z}} \times \overline{\mathbf{u}}' = -\nabla p', \qquad (7)$$

where $-\nabla p'$ is the nondimensionalized pressure gradient force. Equation (7) admits a range of balanced flows, including flows in cyclogeostrophic balance, relevant for considering vortex flows, as in section 3. The nondimensionalized equations governing the horizontal Ekman flow can then be written in vector form as

$$\varepsilon \mathbf{u}_{e}' \cdot \nabla \overline{\mathbf{u}}' + \varepsilon \frac{\delta U_{e}}{U_{e}} \overline{\mathbf{u}}' \cdot \nabla \mathbf{u}_{e}' + \varepsilon_{e} \frac{\delta U_{e}}{U_{e}} \mathbf{u}_{e}' \cdot \nabla \mathbf{u}_{e}' + \hat{\mathbf{z}} \times \mathbf{u}_{e}' = \frac{\partial \tau'}{\partial z'},$$
(8)

where the gradient in Ekman flow is scaled as $\nabla u_e \sim \delta U_e/L$. For a spatially uniform wind stress $\delta U_e/U_e \sim \varepsilon$ (Stern 1965; Klein and Hua 1988); hence, the second and third terms on the left-hand side appear at $O(\varepsilon^2)$ and $O(\varepsilon_e \varepsilon)$, respectively.



FIG. 1. Schematic of the balanced natural coordinate system. The heavy black line denotes a streamline of \bar{u} . Red lines denote examples of the locally tangential (\hat{s}) and normal (\hat{n}) basis vectors, and the vertical dimension (\hat{z}) is directed out of the page. An example osculating circle for a point along the streamline is shown, along with the local radius of curvature *R* (note that for this example, *R* < 0). An example angle between the streamline and the \hat{x} direction is also shown, denoted as φ .

In this article, we consider the same limit as Niiler (1969):

$$\varepsilon_{\rho} \ll 1 \quad \text{and} \quad \varepsilon < 1, \tag{9}$$

but without restriction on the flow geometry, such that only the third term on the left-hand side of (8), representing self-advection by the Ekman flow, is neglected. Considering a typical midlatitude Ekman flow, where $U_e \sim 0.05 \,\mathrm{m \, s^{-1}}$ (using $\tau_o \sim 0.1 \,\mathrm{N \, m^{-2}}$, $h_e \sim 20 \,\mathrm{m}$, and f = $1 \times 10^{-4} \text{s}^{-1}$), it can be seen that for L > -5 km the condition of $\varepsilon_e \ll 1$ will be satisfied. In the case that the balanced Rossby number is also small ($\varepsilon \ll 1$) solutions found using (9) are asymptotically equivalent to the limiting case considered by Stern (1965), as discussed in section 4. However, many important oceanic flows have $\varepsilon \sim 1$, including western boundary currents, flows at low latitude, and submesoscale currents and vortices, and hence will violate the assumptions of Stern (1965). The criteria of (9) are therefore of wider applicability than (1), and the utility of this limit is demonstrated in comparison with a full numerical model in section 3.

To proceed, it is useful to switch to a balanced natural coordinate system (e.g., Holton 2004, his section 3.2). Parameterizing the position of a streamline of the balanced flow by a coordinate *s*, it is possible to define a coordinate system such that \hat{s} is the unit tangent vector in the local streamwise direction, $\hat{\mathbf{n}}$ is the spanwise unit vector normal to the local streamwise direction (defined positive to the left of the balanced flow), and $\hat{\mathbf{z}}$ is the standard unit vector in the vertical (Fig. 1). The total velocity can then be written as

$$\mathbf{u} = (\overline{u} + u_{\rho})\hat{\mathbf{s}} + v_{\rho}\hat{\mathbf{n}} + w_{\rho}\hat{\mathbf{z}}, \qquad (10)$$

where $\overline{u} \equiv |\overline{\mathbf{u}}|$ and subscript *e* denotes the Ekman components projected on the (*s*, *n*, *z*) coordinate system. Further details of this coordinate system are discussed in

the appendix. For simplicity we also assume that \overline{u} is invariant in the streamwise direction, such that $\partial \overline{u}/\partial s = 0$.

In this coordinate system, after dropping primes for simplicity, the relevant equations governing the Ekman flow reduce to¹

$$\varepsilon \overline{u} \frac{\partial v_e}{\partial s} + (1 + \varepsilon 2\Omega) u_e = \frac{\partial \tau_n}{\partial z}, \text{ and } (11)$$

$$\varepsilon \overline{u} \frac{\partial u_e}{\partial s} - (1 + \varepsilon \zeta) v_e = \frac{\partial \tau_s}{\partial z}, \qquad (12)$$

where $\Omega \equiv \overline{u}k$ is the angular velocity, $\zeta \equiv -\partial \overline{u}/\partial n + \Omega$ is the relative vorticity, $(\tau_s, \tau_n) = (\tau \cdot \hat{\mathbf{s}}, \tau \cdot \hat{\mathbf{n}})$, and $k \equiv (\partial \hat{\mathbf{s}}/\partial s) \cdot \hat{\mathbf{n}} = 1/R$ with *R* as the radius of the local osculating circle, defined as a positive value (negative) when streamlines curve to the left (right) of the local balanced flow (Fig. 1). We also note that the relative vorticity consists of two terms: the shear vorticity $-\partial \overline{u}/\partial n$ and the curvature vorticity Ω (Bleck 1991).

The Ekman transport,

$$(M_s, M_n) = \left(\int u_e \, dz, \, \int v_e \, dz\right),\tag{13}$$

can be found by vertically integrating (11) and (12), which results in two coupled ordinary differential equations in the along-flow coordinate:

$$\varepsilon \overline{u} \frac{\partial M_n}{\partial s} + (1 + \varepsilon 2\Omega) M_s = \tau_n, \text{ and } (14)$$

$$\varepsilon \overline{u} \frac{\partial M_s}{\partial s} - (1 + \varepsilon \zeta) M_n = \tau_s.$$
(15)

For arbitrary flow curvature, k is a function of s, and hence the coupled ODEs will have nonconstant coefficients, and solutions can be found numerically. However, there are two additional cases that admit analytical solutions that give further insight into the dynamics: a circular vortex and a weakly nonlinear jet.

3. Circular vortex

a. Analytical solutions

In the case where the balanced flow is circular (k is constant), solutions exist for (14) and (15). It is useful to rewrite the governing equations in polar coordinates,

defined by an azimuthal angle θ and radial direction **r**, defined positive outwards. This transformation is straightforward, noting that $\partial s = r\partial \theta$, where *r* is the vortex radius, and $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ if the azimuthal velocity is positive (cyclonic flow) or $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ if the azimuthal velocity is negative (anticyclonic flow). Polar coordinate counterparts to (14) and (15) are thus given by

$$\varepsilon \Omega \frac{\partial M_r}{\partial \theta} - (1 + \varepsilon 2 \Omega) M_{\theta} = \tau_r, \quad \text{and}$$
 (16)

$$\varepsilon \Omega \frac{\partial M_{\theta}}{\partial \theta} + (1 + \varepsilon \zeta) M_r = \tau_{\theta}, \qquad (17)$$

where M_{θ} is the tangential transport, M_r is the radial transport, and $\zeta \equiv (1/r)\partial(r\overline{u})/\partial r$. These equations, and the results of this section, can be contrasted to Wu and Blumen (1982), who treat the case of an atmospheric vortex under the geostrophic momentum approximation, ignoring advection of the ageostrophic flow. Assuming the wind stress is purely zonal, that is, $\tau_{\theta} = -\sin\theta$, and $\tau_r = \cos\theta$, the Ekman transport for the circular vortex is given by

$$M_r = -\frac{(1 + \varepsilon 3\Omega)}{[(1 + \varepsilon 2\Omega)(1 + \varepsilon \zeta) - \varepsilon^2 \Omega^2]} \sin\theta \quad \text{and} \quad (18)$$

$$M_{\theta} = -\frac{\left[1 + \varepsilon(\Omega + \zeta)\right]}{\left[(1 + \varepsilon 2\Omega)(1 + \varepsilon \zeta) - \varepsilon^2 \Omega^2\right]} \cos\theta,$$
(19)

which in Cartesian coordinate components is

$$M_{x} = \varepsilon \frac{(\zeta - 2\Omega)}{[(1 + \varepsilon 2\Omega)(1 + \varepsilon \zeta) - \varepsilon^{2}\Omega^{2}]} \sin\theta \cos\theta, \quad (20)$$

and

$$M_{y} = -\frac{(1 + \epsilon\Omega + \epsilon2\Omega\sin^{2}\theta + \epsilon\zeta\cos^{2}\theta)}{[(1 + \epsilon2\Omega)(1 + \epsilon\zeta) - \epsilon^{2}\Omega^{2}]}.$$
 (21)

Note that except in the special case of a vortex in solid body rotation (where $\zeta = 2\Omega$), there can be a nonzero component of the Ekman transport in the direction of the zonal wind stress, so that contrary to classical Ekman theory the Ekman transport is not purely perpendicular to the wind. The Ekman pumping velocity, w_e , accurate to higher order in ε than (2), can be found by taking the horizontal divergence of (20) and (21).

In the limit of the small Rossby number $\varepsilon^2 \ll 1$, (18)–(21) become

$$M_r \approx -[1 - \varepsilon(\zeta - \Omega)]\sin\theta,$$
 (22)

$$M_{\theta} \approx -(1 - \epsilon \Omega) \cos\theta, \qquad (23)$$

$$M_{x} \approx \varepsilon(\zeta - 2\Omega) \sin\theta \cos\theta$$
, and (24)

$$M_{v} \approx -\{1 - \varepsilon[(\zeta - \Omega)\sin^{2}\theta + \Omega\cos^{2}\theta]\}.$$
 (25)

¹Note that for a wind stress that is uniform everywhere, in the balanced-natural coordinate system $\partial(\tau \cdot \hat{\mathbf{n}})/\partial s = k(\tau \cdot \hat{\mathbf{n}})$, and $\partial(\tau \cdot \hat{\mathbf{n}})/\partial s = -k(\tau \cdot \hat{\mathbf{s}})$. Hence, $\partial U_e/U_e \sim 1$, and we therefore do not include these additional scaling factors, except where necessary for clarity.

The above solutions fully determine the Ekman transport and hence can be used to give an equation for the unknown vector potential **A** in (4), which nondimensionally can be shown to be $\mathbf{A}' = \varepsilon \overline{u}' \cos\theta \hat{\mathbf{z}}$.

For a straight current (i.e., without curvature) oriented parallel to the wind, with Rossby number ε_0 and nondimensional vertical vorticity ζ_0 , Thomas and Rhines (2002) showed that for small ε_0 the magnitude of the nondimensional Ekman transport is

$$|M_0| \approx 1 - \varepsilon_0 \zeta_0. \tag{26}$$

Evaluating (25) where the wind is parallel to the flow gives

$$|M_{\nu}| \approx 1 - \varepsilon(\zeta - \Omega) \quad \text{at} \quad \theta = \pm \pi/2$$
 (27)

which in comparison with (26) shows that the two expressions differ by a term proportional to the angular velocity of the vortex. This term exactly cancels the curvature vorticity, showing that to leading order the curved nonlinear Ekman balance for a wind aligned with the vortex flow is affected only by the shear vorticity $\zeta - \Omega$ rather than the total flow vorticity ζ . Consequently, if one were to use (26) to calculate the Ekman transport in a circular vortex, the resulting transport would be overestimated in an anticyclone and underestimated in a cyclone. Where the wind is perpendicular to the flow, (25) becomes

$$|M_{\nu}| \approx 1 - \varepsilon \Omega \quad \text{at} \quad \theta = 0 \quad \text{or} \quad \pi,$$
 (28)

highlighting how the Ekman transport only involves the curvature vorticity in these sectors of the vortex. The physical explanation for the dependence of the Ekman transport on the shear and curvature vorticity for the two wind alignments is illustrated in Fig. 2 and described below.

The physics can be understood in terms of the vortex dynamics of the Ekman flow, which is governed by the vorticity equation for a steady flow with a barotropic pressure gradient and friction:

$$0 = \boldsymbol{\omega}_a \cdot \nabla \mathbf{u} + \nabla \times \mathbf{F}, \tag{29}$$

where $\boldsymbol{\omega}_a = \nabla \times \mathbf{u} + f\hat{\mathbf{z}}$ is the absolute vorticity that can be tilted and stretched via the first term in (29), and **F** is the frictional force that generates vorticity by exerting a torque on the fluid. Considering only the meridional component of (29) and simplifying using the scaling given by (9) gives

$$\underbrace{-\frac{\partial v_e}{\partial z} \frac{\partial \overline{v}}{\partial x}}_{\text{HTILT}} + \underbrace{\frac{\partial u_e}{\partial z} \frac{\partial \overline{v}}{\partial y}}_{\text{STRETCH}} + \underbrace{(f + \zeta) \frac{\partial v_e}{\partial z}}_{\text{VTILT}} \approx -\frac{\partial F^x}{\partial z}.$$
 (30)

In the classical Ekman vorticity balance the frictional torque is exactly cancelled by the tilting of the vertical component of the absolute vorticity VTILT (e.g., Fig. 2a, VTILT) and tilting and stretching of vorticity in the horizontal direction is neglected. This is not justified for a current with curvature. The vertical shear of the Ekman flow results in a horizontal component of the vorticity that runs parallel to the zonal wind stress $\xi_{\rm DOWN} = -\partial v_e/\partial z$. Where the wind is aligned with the balanced flow, the curved current tilts ξ_{DOWN} at a rate HTILT = $\xi_{\text{DOWN}} \partial \overline{v} / \partial x$ (e.g., Fig. 2b, HTILT), canceling the term in VTILT associated with the curvature vorticity. Here, $\overline{v}(y) = 0$, and the vortex stretching is zero. As a result, the Ekman transport at this location in the vortex [(27)] depends only on the shear vorticity. Where the winds are perpendicular to the vortex flow, it is the horizontal shear rather than the curvature of the balanced flow that tilts ξ_{DOWN} and partially cancels VTILT (e.g., Fig. 2c; HTILT). At these locations \overline{v} is at its maximum, or minimum, value, and hence again STRETCH = 0. Consequently, the Ekman transport (28) varies only with the curvature vorticity for winds normal to the flow.

b. Example vortices

An illustration of the effect the above has on the patterns of Ekman transport, and pumping, in a circular vortex is shown in Fig. 3. The eddy structure is consistent with a Gaussian sea surface height perturbation, with parameters chosen such that $U \sim 0.16 \,\mathrm{m \, s^{-1}}$, $R = 75 \,\mathrm{km}$, $f = 7.27 \times 10^{-5} \text{ s}^{-1}$, and $\varepsilon \sim 0.03$, consistent with the observed global-mean properties of midlatitude mesoscale eddies (Gaube et al. 2015). For a uniform zonal wind stress, the zonal transport develops a quadrupole pattern, emphasizing that the nonlinear Ekman transport is not strictly perpendicular to the wind stress. The meridional transport converges (diverges) on the north (south) side of the cyclonic vortex, with the pattern reversed for the vortex with anticyclonic flow and with slight differences in structure between the two cases. These patterns in Ekman transport lead to a dipole of vertical velocity across a vortex, with vertical velocity magnitudes enhanced (reduced) at $O(\varepsilon^2)$ in the cyclonic (anticyclonic) case relative to the solution of Stern (1965).

The theory also suggests that the nonlinear Ekman transport, and pumping, will be sensitive to the particular velocity structure of the vortex, contrary to solutions that depend only on ζ . An example of this is shown in Fig. 4 for an anticyclonic submesoscale vortex. The balanced velocity is assumed to be in cyclogeostrophic balance, with parameters such that the maximum balanced velocity is ~0.25 m s⁻¹, R = 12 km, $f = 10^{-4}$ s⁻¹, $\tau_o = 0.1$ N m⁻², and $\varepsilon \sim 0.2$ (McWilliams 1985). The change in the velocity structure of the vortex by the centrifugal acceleration term in the cyclogeostrophic



FIG. 2. Schematic of the vorticity dynamics of the Ekman transport at a circular vortex forced by a uniform wind stress in the positive x direction (τ , green vectors). (a) The Ekman spiral (blue vectors) has vorticity in the downwind ($\xi_{DOWN} = -\partial v_e/\partial z$) and crosswind ($\xi_{CROSS} = \partial u_e/\partial z$) directions. In the classical nonlinear Ekman vorticity balance, the frictional input of vorticity by the wind is balanced by the tilting of the vertical component of the balanced flow's absolute vorticity $f + \zeta$ by the Ekman shear VTILT. (b) However, in a curved current, tilting of the vorticity dynamics. The balanced flow's curvature vorticity is responsible for the tilting of ξ_{DOWN} where the wind stress aligns with the balanced flow. (c) Where the wind is perpendicular to the current it is the shear vorticity that tilts ξ_{DOWN} . In both cases, HTILT tends to oppose VTILT, and consequently the Ekman transport varies inversely with the shear and curvature vorticity in the two regions, respectively.

balance leads to different distribution of shear and curvature vorticity across the eddy than for the example mesoscale eddy shown in Fig. 3. Through (18) and (19) this leads to changes in both horizontal transport components, an example of which is the enhancement of the meridional transport on both the upwind and downwind sides of the vortex core (Fig. 4b). This changes the vertical velocity field, including shifting the location of maximum Ekman pumping velocities in toward the vortex center (Fig. 4c), which is not captured by (2) (Fig. 4d).

To provide further validation of the theory, we run the MITgcm (Marshall et al. 1997) in a doubly periodic domain of 300-m depth ($\Delta x = \Delta y = 2.5 \text{ km}, \Delta z = 3 \text{ m}$), with no vertical stratification, and a uniform vertical viscosity of $\nu = 10^{-2} \text{ m}^2 \text{ s}^{-1}$. The model is initialized with a positive Gaussian sea surface height perturbation and an anticyclonic barotropic balanced velocity field, with parameters representative of a midlatitude

mesoscale eddy, as given above for Fig. 3. A uniform zonal surface wind stress is increased slowly in time, to minimize transients, reaching a maximum value of $\tau_o = 0.1 \,\mathrm{Nm^{-2}}$. Ageostrophic velocity components are calculated from the model output as $\mathbf{u}_e(z) =$ $\mathbf{u}(z) - \mathbf{u}(z = -300 \,\mathrm{m})$ and averaged over the last five inertial periods of the model integration. The Ekman Rossby number, calculated using the maximum modeled ageostrophic velocity, is thus $\varepsilon_e \sim 0.018$.

Model results confirm the above theoretical analysis, as shown in Fig. 5. Specifically, the theory given here correctly predicts the patterns of horizontal transport, including the quadrupole pattern aligned with the surface wind stress. Vertical velocity calculated from the divergence of (20) and (21) also more accurately reproduces the modeled vertical velocity field than (2). It is notable that the theory captures the numerical solution well given that for these parameters, $\varepsilon/\varepsilon_e \sim 2$, suggesting the potential relative importance of the



FIG. 3. Ekman transports and vertical velocities for a circular vortex forced by a westerly wind stress. The balanced eddy velocity has radial structure as shown in the top row (see legend), chosen to be consistent with a Gaussian sea surface height profile with length scale R = 75 km. From the second row down to the bottom, the rows show zonal transport M_x , meridional transport M_y , vertical velocity w_e , and the difference between w_e and the vertical velocity calculated using (2) w'. Color scales are normalized as indicated in the row labels, and length scales are normalized by the eddy length scale R. Dashed lines indicate radial positions of 1R and 2R for each vortex, and parameter values are discussed in section 3b.



FIG. 4. Ekman transports and vertical velocities for an anticyclonic submesoscale circular vortex, forced by a uniform westerly wind stress. (a) Zonal transport M_x , (b) meridional transport M_y , (c) vertical velocity w_e , and (d) the difference between w_e and the vertical velocity calculated using (2) w'. Color scales are normalized as indicated in the plot titles, using a definition of the Rossby number based on the extreme value of the relative vorticity, $\varepsilon \sim \max(|\zeta|)/f \approx 0.6$. Dashed lines indicate radial positions of 1R and 2R for each vortex, with R = 12 km, as discussed in section 3b.

nonlinear Ekman self-advection terms in (8), which are neglected in the theory considered here.

4. Flow with arbitrary curvature

In this section, we consider an approximate solution to (14) and (15) to demonstrate that the results of section 3 extend to flows of other geometries. These approximate solutions are shown to give the appropriate horizontal transports for use with (2). Examples of the Ekman pumping field over a meandering jet are then calculated numerically, demonstrating the importance of properly accounting for curvature effects.

a. Approximate solutions for $\varepsilon \ll 1$

To form an approximate solution to (14) and (15), note that the first terms on the left-hand side represent

the $O(\varepsilon)$ streamwise advection of the ageostrophic flow by the balanced flow. If it is assumed that the balanced Rossby number is sufficiently small ($\varepsilon \ll 1$), it is possible to find approximate analytical solutions that retain the advection of the O(1) ageostrophic flow but ignore higher-order advective terms, an assumption of weak nonlinearity. To do this, note that (14) and (15) can be written as

$$M_n \approx -\tau_s + O(\varepsilon)$$
, and (31)

$$M_{\rm s} \approx \tau_{\rm n} + O(\varepsilon). \tag{32}$$

Retaining only terms of $O(\varepsilon)$ gives

$$-\varepsilon \overline{u} \frac{\partial \tau_s}{\partial s} + (1 + \varepsilon 2\Omega) M_s \approx \tau_n, \quad \text{and} \qquad (33)$$



FIG. 5. Results from a full primitive equation numerical model, with parameters consistent with Fig. 3, as discussed in section 3b. (a) Modeled zonal Ekman transport. (b) Modeled meridional Ekman transport. (c) Modeled vertical Ekman pumping velocity. (d) Difference between vertical velocities from the numerical model, w_e^{NUM} , and those derived using the solution of Stern (1965) [(2)]. (e) As in (d), but for the vertical velocities calculated as the horizontal divergence of the Ekman transport, with components (20) and (21). Values in each plot are normalized as indicated in the titles. The center of the vortex in (e) is blanked due to spurious numerical errors associated with the finite grid discretization.

$$\varepsilon \overline{u} \frac{\partial \tau_n}{\partial s} - (1 + \varepsilon \zeta) M_n \approx \tau_s.$$
(34)

Hence, the approximate solutions for the Ekman transport with arbitrary curvature can be written as

$$M_s \approx (1 - \varepsilon \Omega) \tau_n$$
, and (35)

$$M_n \approx -[1 - \varepsilon(\zeta - \Omega)]\tau_s. \tag{36}$$

These transport relationships also remain valid for spatially varying winds if the wind stress magnitude varies over a length scale L_{τ} , such that $L/L_{\tau} \leq \varepsilon$, a

requirement that will frequently be satisfied for typical ocean and atmosphere conditions. It should also be noted that the transformation to balanced natural coordinates is not Galilean invariant (Viúdez and Haney 1996), and in particular the decomposition of the total vorticity (which is Galilean invariant) into the shear and curvature vorticity components depends on the geometry and magnitude of the balanced flow. However, writing the resulting Ekman transport in terms of the shear and curvature vorticity provides a compact way to represent the transport for a variety of flow geometries. To convert to Cartesian coordinates, define $\varphi \equiv \cos^{-1}(\hat{\mathbf{s}} \cdot \hat{\mathbf{x}})$, the angle that the $\hat{\mathbf{s}}$ vector makes with the zonal (Fig. 1). Then, assuming for simplicity that the wind stress is purely zonal, $\tau_s = \cos\varphi$ and $\tau_n = -\sin\varphi$, such that

$$M_{r} \approx -\varepsilon(\zeta - 2\Omega) \sin\varphi \cos\varphi$$
, and (37)

$$M_{\nu} \approx -\{1 - \varepsilon [(\zeta - \Omega)\cos^2 \varphi + \Omega \sin^2 \varphi]\}.$$
 (38)

Equations (35)–(38) are the generalized equivalents of (22)–(25) for a weakly nonlinear flow with arbitrary curvature. Hence, the modifications of the Ekman transport due to the arbitrary flow curvature, and the physical mechanisms involved, are the same as those discussed above in relation to the circular vortex. Notably, as was the case for the circular vortex, but contrary to classical linear Ekman theory, the nonlinear Ekman transport again has a component parallel to the wind stress: $(M_s, M_n) \cdot (\tau_s, \tau_n) \approx \varepsilon(\zeta - 2\Omega)\tau_s\tau_n$.

The Ekman pumping velocity can be found by taking the horizontal divergence

$$w_e = \frac{\partial M_s}{\partial s} + \frac{\partial M_n}{\partial n} - M_n k, \qquad (39)$$

and using (35) and (36), which gives

$$w_e \approx \varepsilon \sin \varphi \frac{\partial \zeta}{\partial s} + \varepsilon \cos \varphi \frac{\partial \zeta}{\partial n} = \varepsilon \frac{\partial \zeta}{\partial y},$$
 (40)

which is asymptotically equivalent to (2). These approximate solutions thus extend the result of Stern (1965) by providing the correct expressions for the horizontal Ekman transport components (4), which, in contrast to Niiler (1969), are valid in balanced flows with curvature.

b. Numerical solutions for a meandering jet

To further illustrate the effects of curving flows on Ekman dynamics, we calculate numerical solutions for w_e at a velocity jet experiencing increasing amplitude sinusoidal meanders,² with a fixed Gaussian across-front velocity profile, as shown in Fig. 6. We note that the governing equations (14) and (15) support an oscillatory



FIG. 6. Vertical Ekman velocities at a jet with increasing amplitude sinusoidal meanders (as discussed in section 4), with normalized meander amplitudes labeled in each plot. The cross-frontal velocity structure is shown by the dashed black line in the top panel, and the solid black lines indicate the position of the jet velocity core (where the velocity is equal to u_{max}) as well as $e^{-1}\overline{u}_{max}$. Length scales are nondimensionalized by the alongfront meander wavelength (120 km), chosen to be consistent with a jet being distorted by midlatitude eddies of Rossby radius ~30 km. Vertical velocities are nondimensionalized by $\varepsilon \tau_o \sqrt{2}/(\rho fL)$, where L = 10 km is the across-front length scale, $f = 10^{-4} \text{ s}^{-1}$, and $\overline{u}_{max} = 0.5 \text{ m s}^{-1}$, such that $\varepsilon = 0.5$. In all cases the wind stress is purely zonal.

mode that arises in the homogeneous solution to the equations (section 5); therefore, to suppress these oscillations and emphasize the particular solution, we add a small linear damping $-0.1f(M_n, M_s)$ to the right-hand side of (14) and (15), respectively. We then solve the coupled equations using a Runge–Kutta method, with a Dirichlet boundary condition of $M_{s,n} = 0$ at s = 0. Initially, at s = 0, the balanced flow is purely zonal, and there is no wind stress applied. As s increases, wind stress is increased to generate the Ekman transport, and farther downstream, the amplitude of the frontal meanders is increased. This solution method is directly analogous to how similar problems are often solved in the time domain; however, here the equations are integrated in the s coordinate.

The calculated vertical velocity fields display many of the same characteristics discussed in relation to the circular vortex, with w_e changing signs on either side of the jet core and intensification of w_e near the crests of the

0.5

² To simultaneously satisfy the conditions of nondivergent balanced flow, and $\partial \overline{u}/\partial s = 0$, streamlines must not be diffuent, and hence must maintain a constant offset distance, α . For a given parameterized curve y = f(x), the position of an offset curve, (x_o, y_o) , can be found using $x_o = x - \alpha y'/\sqrt{1 + y'^2}$ and $y_o = y + \alpha/\sqrt{1 + y'^2}$, where primes indicate differentiation with respect to the x direction. Here we refer to the shape of a meander by the shape of the base curve, y = f(x).



FIG. 7. (top) Normalized Ekman vertical velocity $\{\widehat{w_e} = w_e [\varepsilon \tau_o \sqrt{2}/(\rho fL)]^{-1}\}$ from the numerical solutions discussed in section 4. (middle) Normalized Ekman pumping vertical velocity from (2) $\{\widehat{w_{ST}} = w_{STERN} [\varepsilon \tau_o \sqrt{2}/(\rho fL)]^{-1}\}$. (bottom) Normalized difference between the two solutions $\{\widehat{w'} = (w_e - w_{STERN})[\varepsilon^2 \tau_o/(fR)]^{-1}\}$, where R = 59 km is the minimum radius of curvature for the sinusoidal base curve}. The case shown corresponds to the third panel ($\widehat{A} = 0.05$) of Fig. 6, with parameters as discussed in the caption of that figure.

meanders where Ω is most negative. These solutions can be compared to the vertical velocities calculated using (2) (Fig. 7), showing that the vertical velocities are less strongly dependent on meander phase when $O(\varepsilon^2)$ curvature effects are accounted for. Of additional note are the high-wavenumber features that develop as the meander amplitude increases; these are associated with oscillations in the Ekman transport, which is the subject of the following section.

5. Oscillations, resonance, and instability

The above solutions, which are the principal results of this article, represent the forced, that is, particular, components of the general solutions to (14) and (15). However, it is worth briefly considering the homogenous component of the solutions to demonstrate the existence of oscillations and potential for resonance and instabilities in the Ekman flow that arise due to advection along a curving trajectory.

The homogenous portion of the equation governing the spanwise Ekman flow [(12)] can be written as

$$\frac{\partial^2 v_e}{\partial s^2} - \frac{2\varepsilon}{1 + \varepsilon 2\Omega} \frac{\partial \Omega}{\partial s} \frac{\partial v_e}{\partial s} + \frac{(1 + \varepsilon \zeta)(1 + \varepsilon 2\Omega)}{\varepsilon^2 \overline{u}^2} v_e = 0, \quad (41)$$

which is the equation for a harmonic oscillator with spatially varying coefficients. Thus, $F^2 \equiv (1 + \varepsilon \zeta)(1 + \varepsilon 2\Omega)$ defines the squared frequency of what can be interpreted as Lagrangian inertial oscillations,³ which appear in the advected Ekman flow as elliptical oscillations with the along-flow wavenumber of F/\overline{u} . Changes in the wind stress in the moving frame as a fluid parcel is advected by a curving balanced flow can therefore excite oscillations in the ageostrophic flow, analogous to how temporal changes in wind stress excite inertial oscillations. An alternate physical explanation for this can be seen by considering the Lagrangian equations of motion, where, in the presence of a surface wind stress, fluid parcels will be transported across isolines of balanced flow, leading to accelerations due to the unbalanced component of the pressure gradient force.

An example of the generation of free Lagrangian inertial oscillations is shown in Fig. 8 for a balanced flow that follows a Gaussian meander, solved as discussed in section 4, but without the additional damping terms. For simplicity the wind stress is taken to be purely zonal. Large-amplitude oscillations are apparent in the wake of the Gaussian meander, the dynamics of which can be interpreted in terms of the vertical relative vorticity of the Ekman flow $\zeta_e = \hat{z} \cdot \nabla \times \mathbf{u}_e$, which evolves in the alongfront direction such that

$$\underbrace{\varepsilon \frac{\delta U_e}{U_e} \overline{u} \frac{\partial \zeta_e}{\partial s}}_{\text{ADV}} = \underbrace{(1 + \varepsilon \zeta) \frac{\partial w_e}{\partial z}}_{\text{STRETCH}} - \underbrace{\varepsilon u_e \frac{\partial \zeta}{\partial s} - \varepsilon v_e \frac{\partial \zeta}{\partial n}}_{\text{GRAD}} + \underbrace{\hat{z} \cdot \nabla \times \frac{\partial \tau}{\partial z}}_{\text{FORCE}}.$$
(42)

From left to right, the terms in (42) represent the alongfront advection of ageostrophic vorticity by the balanced flow (ADV), stretching of the absolute background vorticity by the Ekman flow (STRETCH), advection of the gradient in background vorticity by the Ekman flow (GRAD) (similar to the β term in vorticity budgets due to the meridional gradient of f), and the curl of the turbulent Reynolds stress (FORCE) (which for a uniform wind field is zero). The leading-order Ekman velocity field is irrotational for a uniform wind stress, and hence $\delta U_e/U_e \sim \varepsilon$, and the ADV term on the left-hand side appears at $O(\varepsilon^2)$, as in Stern (1965).

Upstream of the meander, the Ekman transport is irrotational, and (42) is a balance between Ekman advection of the gradient in the shear vorticity, and the stretching of

³ The term F^2 is also sometimes termed the absolute centrifugal stability (e.g., Smyth and McWilliams 1998).



FIG. 8. (top) Ekman transport (as indicated by stick vectors) along the core of a Gaussian velocity jet that follows a Gaussian meander in meridional position (as indicated by solid line). Length is normalized by the approximate natural wavelength $\lambda_a = 2\pi i m f$. (middle) Terms in the Ekman vorticity budget, taken as the vertical integral of (42), normalized by $h_e \overline{u} f/L$, where $h_e = 20$ m and $L = 0.3\lambda_n$ is the cross-front scale of the velocity jet. (bottom) Cumulative alongfront integrals of the terms shown above, normalized by $h_e \overline{u} f$. Wind stress is uniform in the zonal direction.

the absolute vorticity by the Ekman vertical velocities. As the flow enters the meander, ageostrophic vorticity is enhanced by the GRAD term, principally through the meridional advection of the gradient in curvature vorticity. Downstream of the meander, the vorticity budget consists of along-flow oscillations in the advection and stretching of vorticity, which characterizes the undamped free Lagrangian inertial oscillations. This can be contrasted with a classic inertial oscillation, which has no associated signal in vorticity. We also note that the amplitude of the oscillations in the ageostrophic vorticity grow secularly in the streamwise coordinate via growth in the meridional wavenumber of the Lagrangian inertial oscillations (which can be approximated as $\sim s\overline{u}^{-1}\partial\zeta/\partial n$, similar to how spatial gradients in the background vorticity field lead to secular growth in time for the meridional wavenumber of inertial oscillations (van Meurs 1998).

If the path of the balanced flow varies periodically, the curvature vorticity, and hence the coefficient F^2 in (41),

will be periodic. This allows for the possibility of both external resonance with the wind stress, which is oscillatory in the Lagrangian frame, and growing instabilities due to parametric resonance (Grimshaw 1993). A full analysis of resonance and instability for the nonlinear Ekman transport problem should relax both the steadystate and small Ekman Rossby number ($\varepsilon_e \ll 1$) assumptions utilized here and hence is beyond the scope of the present work. However, we note that for a sinusoidally meandering jet, in the limit of small meander aspect ratio $(A/\lambda < 1)$, where A is the meander amplitude and λ is the meander wavelength) and weak nonlinearity ($\varepsilon \ll 1$), (41) can be approximated as a Mathieu equation, the stability characteristics of which have been widely studied (e.g., Landau and Lifshitz 1960). From this it can be anticipated that growing instabilities will be found for streamwise wavelength λ_s such that $\lambda_s \approx n\pi/m$, where $m = f/\overline{u}$ is the approximate natural wavenumber of (41) and $n = 1, 2, 3, \dots$ (van der Pol and Strutt 1928). The Floquet multipliers of (41) were also calculated numerically and found to confirm that growing instabilities are possible when the frontal aspect ratio is sufficiently large and an integer number of natural wavelengths fit within twice the streamwise meander wavelength. Importantly, the energy source for parametric oscillations is the balanced flow, and hence parametric oscillations and instabilities represent a mechanism that can extract energy from the balanced flow and drive ageostrophic mass transports, independent of the local wind stress.

As the oscillatory terms in (41) and (42) are $O(\varepsilon^2)$, these effects are most likely to be significant in strong balanced flows. However, the natural wavelength of the oscillations is a function of \overline{u} , which will vary across a jet when $\partial \overline{u} / \partial n \neq 0$, and it can therefore be expected that the conditions for resonance or instability will be satisfied at specific cross-jet positions for many ocean flows. These oscillations can also exist in a periodic domain, such as a circular vortex, if the domain length is an integer multiple of the natural wavelength $\lambda_n = 2\pi \overline{u}/F$ The ratio of the natural wavelength to a typical eddy circumference is given by $\lambda_n/(2\pi R) \sim U/fR = \varepsilon$; hence, for geophysical flows, Lagrangian inertial oscillations may arise even when periodic boundary conditions are imposed on (41). Both forced and parametric resonance will give rise to growing oscillations, which could invalidate the assumption used here of small Ekman Rossby number ($\varepsilon_e \ll 1$), and possible feedbacks between the Ekman flow and the balanced flow will be the subject of future work.

6. Conclusions

In this article, we derived the governing equations for Ekman flow in the limit of weak ageostrophic flow and strong balanced flow. Exact analytical solutions for the Ekman transport in a circular vortex are provided and can be used to calculate the vertical Ekman pumping velocity to a higher order of accuracy in ε than possible with previous formulations. Approximate solutions for the Ekman transport for flows with small Rossby number, but arbitrary geometry, are also given. These solutions consist of both divergent and solenoidal components and are shown to be appropriate for use with the Ekman pumping solution of Stern (1965).

Both the exact transport solutions for the circular vortex and the generalized approximate solutions differ from prior formulations derived under the assumption of straight fronts (Niiler 1969; Thomas and Rhines 2002). These differences arise physically from the balanced flow tilting, in the horizontal plane, the horizontal vorticity associated with the Ekman vertical shear. To leading order, for a wind aligned with the balanced flow, this has the effect of cancelling the tilting of the curvature vorticity. This leads to an Ekman balance that is between the turbulent diffusion of horizontal vorticity and the tilting of planetary vorticity plus the vertical component of the shear vorticity rather than the total vorticity. Conversely, for a wind aligned across the balanced flow, the Ekman transport depends to leading order on the curvature vorticity, with no contribution from the shear vorticity.

The effects of curvature on Ekman dynamics will depend on the geometry and strength of the balanced flow, and the direction of the wind stress relative to the currents. For a wind aligned with (across) the balanced flow, differences between the transport solutions given here and prior formulations using the total vorticity will scale as U/fR(U/fL), where R is the radius of curvature, and L is the spanwise length scale. For many mesoscale eddies and jets these effects can thus be expected to be of similar order as the total relative vorticity (Liu and Rossby 1993; Shearman et al. 2000; Chelton et al. 2011). The effect of retaining terms of higher order in ε on the accuracy of the calculated Ekman pumping velocity can be seen by noting that the nonlinear component of the Ekman pumping velocity is itself proportional to ε . Hence, including terms of $O(\varepsilon^2)$ in the solution leads to an $O(\varepsilon)$ relative improvement in accuracy. For the submesoscale vortex considered in section 3b, retaining terms of $O(\varepsilon^2)$ leads to an approximately 30% increase in accuracy, equivalent to vertical velocities of \sim 4 m day⁻¹. Identifying the effect of curved flow on the Ekman transport and Ekman pumping velocity should therefore be a priority of future observational and numerical work.

In some of the flow configurations considered here, the curvature vorticity changes following the balanced flow, while the shear vorticity is assumed to be uniform in the streamwise direction. In reality there is likely to be exchange along the flow between the shear and curvature vorticity (Chew 1974; Viúdez and Haney 1997), which, along with streamwise variations in the magnitude of the balanced flow, could give rise to systematic patterns in Ekman transport along a meandering current. Similarly, the oscillations and growing instabilities discussed in section 5 may also lead to Ekman velocities that eventually violate the initial assumption of small Ekman Rossby number ε_e , particularly when time dependence is included in the problem formulation. Future work will consider the effect of more realistic flow configurations and feedbacks from the Ekman transport on the evolution of the balanced flow.

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APPENDIX

Notes on the Balanced Natural Coordinate System

In this appendix, we note several useful properties and relationships for the balanced natural coordinate system. Generally, the definition of $(\hat{\mathbf{s}}, \hat{\mathbf{n}}, \hat{\mathbf{z}})$ basis vectors following curved streamlines formally defines a moving frame rather than a true coordinate system. In this case, derivatives should be understood as directional derivatives, which can be evaluated as projections of the local $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ basis vectors on a Cartesian gradient operator, as in Viúdez and Haney (1996). However, the barotropic, nondivergent, balanced flow we consider here is complex lamellar [$\bar{\mathbf{u}} \cdot (\nabla \times \bar{\mathbf{u}}) = 0$], and hence the streamline basis vectors define a true coordinate system (Finnigan 1983; Finnigan et al. 1990). Given this, we can define differential operators directly in terms of the (*s*, *n*, *z*) coordinates. As such, we define the gradient of a scalar χ as

$$\nabla \chi \equiv \hat{\mathbf{s}} \frac{\partial \chi}{\partial s} + \hat{\mathbf{n}} \frac{\partial \chi}{\partial n} + \hat{\mathbf{z}} \frac{\partial \chi}{\partial z}.$$
 (A1)

In the balanced natural coordinate system, care has to be taken to properly account for streamline curvature and, in more general balanced flows than we consider here, the effects of diffluent streamlines. Following the notation given in Kusse and Westwig (2006, their

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appendix F), the divergence of a vector \mathbf{V} can thus be denoted as

$$\nabla \cdot \mathbf{V} = \widehat{g^{j}} \cdot \frac{\partial}{\partial x^{j}} (V^{i} \widehat{g}_{i}), \qquad (A2)$$

where indices follow standard tensor notation, $\hat{\mathbf{g}}$ are the basis vectors, $\partial/\partial x^{j}$ is the *j*th component of the gradient operator, the carat notation indicates unit vectors, and V^{i} is the *i*th component of the vector **V**. It can also be noted that the last term on the right-hand side defines the covariant derivative. Evaluating this for the problem considered here gives

$$\nabla \cdot \mathbf{V} = \frac{\partial V_s}{\partial s} + \frac{\partial V_n}{\partial n} + V_s \frac{\partial \varphi}{\partial n} - V_n \frac{\partial \varphi}{\partial s} + \frac{\partial V_z}{\partial z}, \quad (A3)$$

where subscripts denote the component of the vector in each coordinate direction, and φ is the angle the \hat{s} basis vector makes with the *x* direction, as discussed in section 4. Likewise, the curl operator can be defined (Kusse and Westwig 2006) as

$$\nabla \times \mathbf{V} = \widehat{g^{i}} \frac{\partial}{\partial x^{i}} \times (V^{i} \widehat{g_{i}}).$$
 (A4)

Or, for the vertical component of vorticity, which we consider in section 5,

$$\hat{\mathbf{z}} \cdot (\nabla \times \mathbf{V}) = \frac{\partial V_n}{\partial s} - \frac{\partial V_s}{\partial n} + V_s k.$$
 (A5)

Finally, we note that in this coordinate system mixed derivatives do not commute, which can be seen from evaluating the identity $\nabla \times (\nabla \chi) = 0$, and in particular we utilize the relationship (Bell and Keyser 1993)

$$\frac{\partial}{\partial n}\frac{\partial\chi}{\partial s} = \frac{\partial}{\partial s}\frac{\partial\chi}{\partial n} + \frac{\partial\chi}{\partial s}\frac{\partial\varphi}{\partial s} + \frac{\partial\chi}{\partial n}\frac{\partial\varphi}{\partial n}.$$
 (A6)

Additional discussion of the natural coordinate system can be found in Bell and Keyser (1993), including that

$$\frac{\partial \hat{\mathbf{s}}}{\partial \gamma} = \hat{\mathbf{n}} \frac{\partial \varphi}{\partial \gamma}, \quad \frac{\partial \hat{\mathbf{n}}}{\partial \gamma} = -\hat{\mathbf{s}} \frac{\partial \varphi}{\partial \gamma}, \tag{A7}$$

for γ equal to s or n.

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